JOURNAL OF COMBINATORIAL THEORY, Series B **66**, 75–86 (1996)
ARTICLE NO. 0007

Hamiltonian Decompositions of Cayley Graphs on Abelian Groups of Odd Order

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Received July 28, 1992

Alspach has conjectured that any $2k$-regular connected Cayley graph $cay(A, S)$

generating set of an abelian group A of odd order (where a generating set S of a group G is *minimal* if no proper subset of S can generate G).

1. INTRODUCTION

Let $(A, +)$ be a finite group and S be a subset of A with 0 not in S. The Cayley graph $cay(A, S)$ is defined to be the graph G with $V(G) = A$ and $E(G) = \{xy \mid x, y \in A,\ \ x - y \in S \text{ or } y - x \in S\}$. We say an edge xy in $cay(A, S)$ is generated by $s \in S$ if $x - y = s$ or $y - x = s$ and the subgraph Q of $cay(A, S)$ is generated by s if $E(Q)$ consists of all the edges generated by s.

Remark 1. From the definition, it is clear that any element of S with order 2 generates a 1-factor of $cay(A, S)$ while any element of S with order at least 3 generates a 2-factor of $cay(A, S)$. Furthermore, $cay(A, S)$ is connected if and only if S generates A.

It is known that any connected Cayley graph on a finite abelian group is hamiltonian [9]. In [1], Alspach conjectured that any $2k$-regular connected Cayley graph on a finite abelian group has a hamiltonian decomposition. Bermond, Favaron, and Maheo [4] proved that every 4-regular connected Cayley graph $cay(A, S)$ on a finite abelian group A can be decomposed into two hamiltonian cycles. Liu [8] showed that the conjecture is true provided that either $cay(A, S)$ is $2m$-regular and $S = \{s_1, s_2, ..., s_k\}$ is a generating set of A such that $gcd(ord(s_i), ord(s_j)) = 1$ for $i \neq j$ or $S = \{s_1, s_2, s_3\}$ is a minimal generating set of A with $|A|$ being odd. Here our main purpose is to prove the following theorem which establishes the conjecture for a more general case.

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THEOREM 1. *If A is an abelian group of odd order and $S=\{s_1, s_2, ..., s_k\}$ is a minimal generating set of A, then $cay(A, S)$ has a hamiltonian decomposition.*

By convention, terminology and notation not mentioned here are referred to as in [5] and [7].

2. PRELIMINARY RESULTS

First, we recall a definition and some results.

DEFINITION 1. The *cartesian product* $G=G_1 \times G_2$ has vertex set $V(G)=V(G_1) \times V(G_2)$ and edge set $E(G)=\{(u_1, u_2)(v_1, v_2) \mid u_1=v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2=v_2$ and $u_1 v_1 \in E(G_1)\}$.

THEOREM A (Alspach *et al.* [2]). *If C_i is a cycle for each $i=1, 2, ..., n$, then the cartesian product $G=C_1 \times C_2 \times \cdots \times C_n$ can be decomposed into n hamiltonian cycles.*

THEOREM B (Bermond *et al.* [4]). *Every 4-regular connected Cayley graph $cay(A, S)$ on a finite abelian group A can be decomposed into two hamiltonian cycles.*

THEOREM C (Liu [8]). *Let A be a finite abelian group of odd order and $S=\{s_1, s_2, s_3\}$ be a minimal generating set of A. Then, $cay(A, S)$ has a hamiltonian decomposition.*

The following proposition is Lemma 2.5 in [8].

PROPOSITION 1. *Let A be a finite abelian group which is generated by $S=\{s_1, s_2, ..., s_k\}$, A_1 be the subgroup of A which is generated by $S'=\{s_1, s_2, ..., s_{k-1}\}$ and $J=\langle s_k\rangle$. If $A_1 \cap J=\{0\}$, then $cay(A, S)=cay(A_1, S') \times cay(J, \{s_k\})$.*

From now on, we let $C_1=a_1 a_2 \cdots a_n a_1$ and $C_2=b_1 b_2 \cdots b_m b_1$ be two cycles. By convention, the subscripts of a are expressed modulo n and the subscripts of b are expressed modulo m.

DEFINITION 2. For $0 \leqslant r \leqslant m-1$, the *r-pseudo-cartesian product* of C_1 and C_2, denoted by $C_1 \times_r C_2$, is the graph which is obtained from $C_1 \times C_2$ by replacing the edge set $\{(a_1, b_i)(a_n, b_i) \mid 1 \leqslant i \leqslant m\}$ by the edge set $\{(a_1, b_{i+r})(a_n, b_i) \mid 1 \leqslant i \leqslant m\}$.

From the definition, it is easy to see that $C_1 \times_0 C_2 = C_1 \times C_2 = C_1 \times_m C_2$. For convenience, we call the vertex set $\{(a_i, b_j) \mid 1 \leqslant i \leqslant n\}$ the b_j-row and the vertex set $\{(a_i, b_j) \mid 1 \leqslant j \leqslant m\}$ the a_i-column. Also, we call the edges whose two end-vertices have the same first component *vertical* edges and the edges with different first components *horizontal-type* edges in an r-pseudo-cartesian product.

Remark 2. If $gcd(r, m) = t$ in an r-pseudo-cartesian product $C_1 \times_r C_2$, then the horizontal-type edges form a 2-factor H which consists of t cycles of length $(mn)/t$ and any consecutive t rows of $C_1 \times_r C_2$ are on t different cycles of H.

Before proceeding further we give two simple facts.

Fact 1. If $u_1 u_2 \in E(Q_1)$ and $v_1 v_2 \in E(Q_2)$, where Q_1 and Q_2 are two vertex-disjoint cycles, then $(Q_1 \cup Q_2 - \{u_1 u_2, v_1 v_2\}) \cup \{u_1 v_1, u_2 v_2\}$ is a cycle.

Fact 2. Given a cycle C, let $u_1 u_2$ and $v_1 v_2$ be edges of C which are separated by at least two edges. Then $(C - \{u_1 u_2, v_1 v_2\}) \cup \{u_1 v_1, u_2 v_2\}$ is a 2-factor containing at most two cycles.

For the following discussions, we color all horizontal-type edges in $C_1 \times_r C_2$ by one color, say blue, and all vertical edges by another color, say red.

DEFINITION 3. A *color switching* in $C_1 \times_r C_2$ means that for some $1 \leqslant i \leqslant n-1$ and some $1 \leqslant s \leqslant m$ we interchange the colors between the edge sets $\{(a_i, b_s)(a_i, b_{s+1}), (a_{i+1}, b_s)(a_{i+1}, b_{s+1})\}$ and $\{(a_i, b_s)(a_{i+1}, b_s), (a_i, b_{s+1})(a_{i+1}, b_{s+1})\}$, and we call it the $\{a_i, a_{i+1}, b_s, b_{s+1}\}$-color switching.

The next proposition is Lemma 3.13 in [8] which follows easily from Remark 2 and Fact 1.

PROPOSITION 2. *If* $gcd(r, m) = 2t + 1 \geqslant 3$, *then, by making the color switchings* $1, 2, ..., 2t$ *in* $C_1 \times_r C_2$ *shown in Fig.* 1, *we obtain a blue hamiltonian cycle and connect three red cycles in the* a_j*-columns for* $j = 1, 2, 3$ *to a single red cycle.*

Also, by Remark 2 and Fact 1, the following lemma can be derived easily.

LEMMA 1. *Suppose* $n \geqslant 5$ *and* $gcd(r, m) = 2t + 1 \geqslant 3$. *Then, by making the color switchings in* $C_1 \times_r C_2$ *shown in Fig.* 2, *we obtain a blue hamiltonian cycle and connect the red cycles in the* a_j*-columns for* $1 \leqslant j \leqslant y$ *to a single red cycle, where* $y = 3$ *if* $2t + 1 = 3$, *and* $y = 5$ *otherwise.*

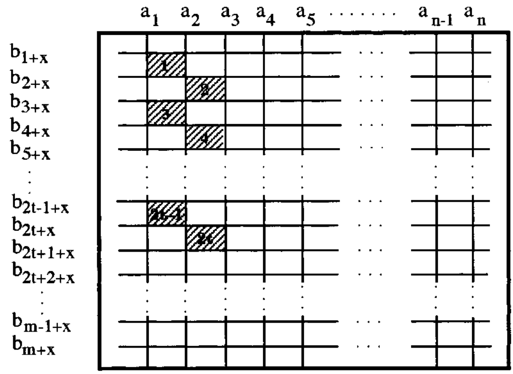

FIG. 1. Color switchings in $C_1 \times_r C_2$, I, where $0 \leqslant x \leqslant m-1$.

LEMMA 2. *If A is an abelian group of odd order and $S = \{s_1, s_2, ..., s_k\}$ is a minimal generating set of A, then $|A| \geqslant 3^k$.*

Proof. We proceed by induction on k. For $k=1$, the result is clear. Assume the result for $k<h$ (with $h \geqslant 2$). Now, we consider $k=h$. Let $S' = \{s_1, s_2, ..., s_{k-1}\}$ and $A' = \langle S' \rangle$. By the induction hypothesis, $|A'| \geqslant 3^{k-1}$. Since S is a minimal generating set of A and $|A|$ is odd, there are at least three cosets of A' in A. Thus, $|A| \geqslant 3\,|A'| \geqslant 3^k$. ∎

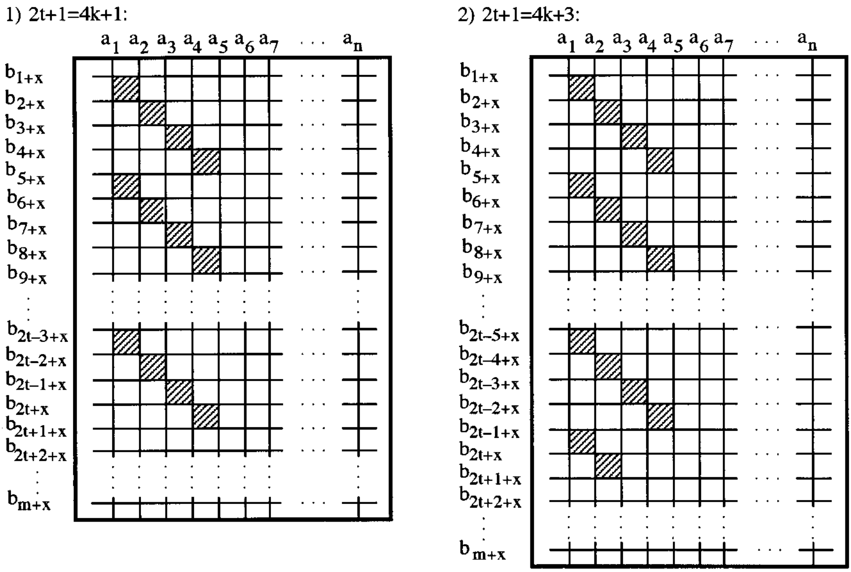

FIG. 2. Color switchings in $C_1 \times_r C_2$, II, where $0 \leqslant x \leqslant m-1$.

LEMMA 3. *If G is a graph of order $n \geqslant 26$ which can be decomposed into three hamiltonian cycles, then, for any hamiltonian decomposition H_1, H_2, H_3 of G, there are three paths $P_1 = u_1 u_2 u_3$, $P_2 = v_1 v_2 v_3 v_4 v_5$, and $P_3 = w_1 w_2 w_3 w_4 w_5$ such that P_i is on H_i for $1 \leqslant i \leqslant 3$, where either P_1 and P_2 have at most one common vertex $u_3 = v_1$ and $V(P_3) \cap [V(P_1) \cup V(P_2)] = \emptyset$ or P_1 and P_3 have at most one common vertex $u_3 = w_1$ and $V(P_2) \cap [V(P_1) \cup V(P_3)] = \emptyset$.*

Proof. Suppose $P_3 = w_1 w_2 w_3 w_4 w_5$ is on H_3. Then the vertices w_j for $1 \leqslant j \leqslant 5$ divide H_2 into five subpaths. Since $n > 25$, at least one of such subpath has length at least 6, so we obtain a path $P_2 = v_1 v_2 v_3 v_4 v_5$ on H_2 such that P_2 and P_3 are vertex-disjoint. Now, the vertices v_i for $1 \leqslant i \leqslant 5$ and w_j for $1 \leqslant j \leqslant 5$ divide H_1 into ten subpaths. If there is one such subpath of length at least 4, then we obtain a path $P_1 = u_1 u_2 u_3$ on H_1 which is vertex-disjoint with P_2 and P_3. Thus we assume that all subpaths have length at most 3. Since $n \geqslant 26$, there are at least six of the subpaths of length 3. The four vertices v_1, v_5, w_1, w_5 are the ends of at least five subpaths on H_1 so that one of them is an end of a subpath of length 3. By reordering the v-path or the w-path if necessary, we conclude that there is a desired subpath $P_1 = u_1 u_2 u_3$ on H_1 with either $v_1 = u_3$ or $w_1 = u_3$. ∎

Similar to the proof of Lemma 3, we can derive the following result.

LEMMA 4. *If G is a $2k$-regular graph of order $n > 5(5k-7)$, where $k \geqslant 2$, and G has a hamiltonian decomposition, then for any hamiltonian decomposition $H_1, H_2, ..., H_k$ of G, there exist k vertex-disjoint paths $P_1 = u_1 v_1 w_1$ and $P_j = u_j v_j w_j x_j y_j$ for $2 \leqslant j \leqslant k$ such that P_j is on H_j for $1 \leqslant j \leqslant k$.*

Proof. If $0 < t \leqslant k-1$ vertex-disjoint subpaths P_j of length 4 have been chosen on the cycles H_j, $k-t+1 \leqslant j \leqslant k$, their $5t$ vertices divide the remaining $n - 5t > 5(5k - t - 7)$ vertices of H_{k-t} into $5t$ segments. The longest one, say P, has at least five vertices since $4(5t) < n - 5t$ if $t < k-1$, and at least three vertices if $t = k-1$ since $2(5t) < n - 5t$ owing to $k \geqslant 2$. Thus we may choose a subpath P_{k-t} of P as desired. ∎

3. PROOF OF THEOREM 1

Throughout this section we let $C_1 = a_1 a_2 \cdots a_n a_1$ and $C_2 = b_1 b_2 \cdots b_m b_1$ be two cycles of order n and m, respectively. The subscripts of a are expressed modulo n and the subscripts of b are expressed modulo m. By convention, we give a natural direction to a drawing of $C_1 \times_r C_2$ on a torus as in Fig. 1 so that the a_i-column is to the left of the a_{i+1}-column for $1 \leqslant i \leqslant n-1$.

Let A be a finite abelian group, $S=\{s_1, s_2, ..., s_k\}$ be a minimal generating set of A and $m=ord(s_k)\geqslant 3$. Let F be the 2-factor of $cay(A, S)$ which is generated by s_k. Then F consists of $n=|A|/m$ cycles of length m. Let $J=\langle s_k\rangle$ and $\bar{S}=\{\bar{s}_1, \bar{s}_2, ..., \bar{s}_{k-1}\}\subseteq A_1=A/J$, where we use $\bar{x}$ to represent the coset $x+J$.

Definition 4. For any edge $\bar{x}\bar{y}$ of $cay(A_1, \bar{S})$, where $\bar{x}-\bar{y}=\bar{s}_i\in\bar{S}$, we call the edge set $\{u_1u_2 \mid \bar{u}_1=\bar{x},\ \bar{u}_2=\bar{y}$ and $u_1-u_2=s_i\}$ of $cay(A, S)$ the *lifting edge* set of the edge $\bar{x}\bar{y}$.

Definition 5. For any subgraph $\bar{Q}$ of $cay(A_1, \bar{S})$, the spanning subgraph Q of $cay(A, S)$ with the edge set being the union of the lifting edge sets of the edges of $\bar{Q}$ is called the *subgraph lifted by* $\bar{Q}$ and we say $\bar{Q}$ lifts to Q.

It is easy to see from the above definitions that edge-disjoint subgraphs of $cay(A_1, \bar{S})$ lift to edge-disjoint subgraphs of $cay(A, S)$.

The following proposition is Lemma 3.7 in [8].

Proposition 3. *Any hamiltonian cycle $\bar{H}=\bar{g}_1\bar{g}_2\cdots\bar{g}_n\bar{g}_1$ of $cay(A_1, \bar{S})$ lifts to a 2-factor H of $cay(A, S)$. Furthermore, under the map Φ: $a_i+b_j\to(a_i, b_j)$, the union of F and H is isomorphic to an r-pseudo-cartesian product $C_1\times_r C_2$ in which the edges of H are horizontal-type and the edges of F are vertical, where $0\leqslant r\leqslant m-1$, $\bar{a}_i=\bar{g}_i$ for $1\leqslant i\leqslant n$, and $b_j=(j-1)\,s_k$ for $1\leqslant j\leqslant m$.*

Next we introduce a special class of graphs which plays a central role in our discussion.

Definition 6. For $k\geqslant 2$, define D_k to be a $2k$-regular graph satisfying:

(1) $V(D_k)=\{(a_i, b_j) \mid 1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m\}$,

(2) $E(D_k)$ can be decomposed into 2-factors $H_1, H_2, ..., H_{k-1}$ and F,

(3) $F=\bigcup_{i=1}^{n} F_i$, where each F_i is the cycle $(a_i, b_1)(a_i, b_2)\cdots(a_i, b_m)(a_i, b_1)$,

(4) $H_1\cup F=C_1\times_{r_1} C_2$, and

(5) for $2\leqslant j\leqslant k-1$, $H_j\cup F=C_1^j\times_{r_j} C_2$, where $0\leqslant r_j\leqslant m-1$ and $C_1^j=a^{(j)}_{\pi_j(1)}a^{(j)}_{\pi_j(2)}\cdots a^{(j)}_{\pi_j(n)}a^{(j)}_{\pi_j(1)}$ with π_j being a permutation of $\{1, 2, ..., n\}$ and $(a_i^{(j)}, b_t)=(a_i, b_{t+h_{i,j}})$ for $1\leqslant i\leqslant n$.

The example shown in Fig. 3 is a graph D_3 with $n=6$, $m=5$, $r_1=1$, $r_2=2$, $\pi_2=(124653)$, and $h_{i,2}\equiv 3i-1 \pmod 5$ for $1\leqslant i\leqslant 6$.

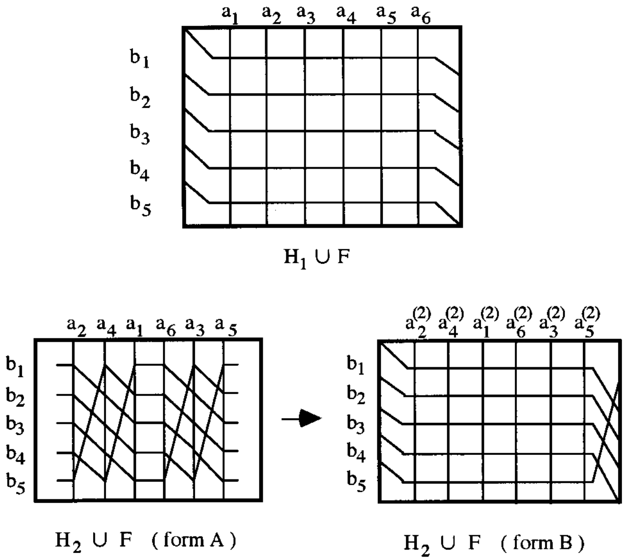

FIG. 3. A graph D_3.

Clearly, from the definition it is easy to see that in each $H_j \cup F = C_1^j \times_{r_j} C_2$ the edges of F are vertical while the edges of H_j are horizontal-type (in the sense that they are not vertical), where $C_1^1 = C_1$. Thus, by Remark 2, H_j is the union of $t_j = gcd(r_j, m)$ cycles of equal length $(mn)/t_j$. For example, both H_1 and H_2 of the graph D_3 in Fig. 3 are hamiltonian cycles.

From Proposition 3 and the definition for D_k the following lemma can be derived easily.

LEMMA 5. *If $cay(A_1, \bar{S})$ can be decomposed into $k-1$ hamiltonian cycles $\bar{H}_j = \bar{g}_{\pi_j(1)} \bar{g}_{\pi_j(2)} \cdots \bar{g}_{\pi_j(n)} \bar{g}_{\pi_j(1)}$ for $1 \leqslant j \leqslant k-1$, where $\pi_1 = I$ (the identity) and π_j is a permutation of $\{1, 2, ..., n\}$ for $2 \leqslant j \leqslant k-1$, then $cay(A, S) \cong D_k$ with each H_j being the 2-factor lifted by $\bar{H}_j$ and F being the 2-factor generated by s_k.*

Proof. By Proposition 3, for each $1 \leqslant j \leqslant k-1$, $\bar{H}_j$ lifts to a 2-factor H_j and the union $H_j \cup F$ is isomorphic to $C_1^j \times_{r_j} C_2$ with $C_1^j = a^{(j)}_{\pi_j(1)} a^{(j)}_{\pi_j(2)} \cdots a^{(j)}_{\pi_j(n)} a^{(j)}_{\pi_j(1)}$ under the map Φ_j: $a_i^{(j)} + b_t \to (a_i^{(j)}, b_t)$, where $\overline{a_i^{(j)}} = \bar{g}_i$ and $b_t = (t-1)\, s_k$. For convenience, we denote $a_i^{(1)} = a_i$ for $1 \leqslant i \leqslant n$, and so $C_1^1 = C_1 = a_1 a_2 \cdots a_n a_1$ and $H_1 \cup F = C_1 \times_{r_1} C_2$. Since $\overline{a_i^{(j)}} = \bar{a}_i$ for $2 \leqslant j \leqslant k-1$ and $1 \leqslant i \leqslant n$, we let $a_i^{(j)} = a_i + h_{i,j} s_k$. Then $a_i^{(j)} + b_t = a_i^{(j)} + (t-1)\, s_k = a_i + (t + h_{i,j} - 1)\, s_k = a_i + b_{t + h_{i,j}}$ and we have $(a_i^{(j)}, b_t) =$

$(a_i, b_{t+h_{i,j}})$. Recall that edge-disjoint subgraphs of $cay(A_1, \bar{S})$ lift to edge-disjoint subgraphs of $cay(A, S)$. We conclude that $cay(A, S) \cong D_k$. ∎

By Lemma 5, we see that an important step to prove Theorem 1 by induction on k is to prove that D_k can be decomposed into k hamiltonian cycles. Thus, our next major effort is to show that with certain restrictions D_k has a hamiltonian decomposition.

For the following discussions, we color the edges of D_k so that all edges of F are of red color and for $1 \leqslant j \leqslant k-1$, all edges of H_j are of color c_j. In order to decompose D_k into hamiltonian cycles, our main idea here is to find some edge-disjoint color switchings so that making those color switchings results in k monochromatic hamiltonian cycles. In each of the following lemmas, we will find the necessary color switchings in two steps: First, we make some color switchings in each $H_j \cup F = C_1^j \times_{r_j} C_2$ to obtain a hamiltonian cycle of color c_j for $1 \leqslant j \leqslant k-1$; then, we concentrate on $H_1 \cup F = C_1 \times_{r_1} C_2$ and make some additional color switchings between red edges and edges of color c_1 to obtain a red hamiltonian cycle meanwhile the edges of color c_1 still form a hamiltonian cycle.

Also, for the following discussions it is very helpful to visualize $H_j \cup F = C_1^j \times_{r_j} C_2$ in a way similar to (form B) of Fig. 3.

LEMMA 6. *Let $m > 0$ and $n > 0$ be odd integers, and let $k \geqslant 2$. Assume each H_j in D_k consists of t_j cycles with $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{k-1}$. If $2 \leqslant t_1 < m$ and the sets $K_j = \{\pi_j(i) \mid 1 \leqslant i \leqslant 3\}$ for $1 \leqslant j \leqslant k-1$ are mutually disjoint, where $\pi_1 = I$, then D_k has a hamiltonian decomposition.*

Proof. Recall that in each $H_j \cup F = C_1^j \times_{r_j} C_2$, the vertical edges are red, the horizontal-type edges are of color c_j, and each row (in the sense that we visualize $H_j \cup F$ as in (form B) of Fig. 3) is in the same cycle of H_j. By Remark 2, each H_j consists of $t_j = gcd(r_j, m) = 2t_j' + 1$ cycles. If $t_j = 1$ for some j, then H_j is already a hamiltonian cycle, so we can disregard it and work on the remaining graph obtained by removing the edges of H_j from D_k. Thus, we may assume that $t_j = 2t_j' + 1 \geqslant 3$ for $1 \leqslant j \leqslant k-1$. Since $t_1 \mid m$, $t_1 < m$, and m is odd, we must have $t_1 \leqslant m/3$, implying that $t_i \leqslant t_1 \leqslant m/3$ for $2 \leqslant i \leqslant k-1$. Next, we first apply Proposition 2 to $H_1 \cup F = C_1 \times_{r_1} C_2$ with $x = 0$ to obtain a hamiltonian cycle of color c_1 and connect the red cycles in the a_j-columns for $1 \leqslant j \leqslant 3$ to a single red cycle. Clearly, the vertical edge $e = (a_3, b_1)(a_3, b_2)$ is still red. Note that for $2 \leqslant j \leqslant k-1$, when we apply Proposition 2 to $H_j \cup F = C_1^j \times_{r_j} C_2$, the vertical edges in the $a_{\pi_j(1)}$-column which change color are on a subpath of order $y \leqslant m/3 - 1$ of the red cycle in that column and the same is true in the $a_{\pi_j(3)}$-column while the vertical edges in the $a_{\pi_j(2)}$-column which change color form a subpath of order $y \leqslant m/3$ of the corresponding red cycle. Since the sets $K_1, K_2, \ldots, K_{k-1}$ are mutually disjoint, we conclude that for each $2 \leqslant j \leqslant k-1$, by choosing x properly, we can apply Proposition 2 to $H_j \cup F = C_1^j \times_{r_j} C_2$ without using any vertical edges from the paths $P_i = (a_i, b_1)(a_i, b_2)(a_i, b_3)$ for $4 \leqslant i \leqslant n$ in $H_1 \cup F = C_1 \times_{r_1} C_2$ to obtain a hamiltonian cycle of color c_j and connect the red cycles in the $a_{\pi_j(i)}$-columns for $1 \leqslant i \leqslant 3$ to a single red cycle. Clearly, we now have $n-2(k-1)$ red cycles and each column is in a single red cycle. Since n is odd, $n-2(k-1)=2n_1+1$ for some $n_1>0$. Now, we will find some additional edge-disjoint color switchings between red edges and edges of color c_1 so that we can connect all red cycles to a red hamiltonian cycle while we still have a hamiltonian cycle of color c_1. For that purpose, we focus on $H_1 \cup F = C_1 \times_{r_1} C_2$. Let $0 < z_1 < z_2 < \cdots < z_{2n_1}$ be the integer sequence defined recursively as follows. Let $z_1 = 3$ and $R = \{1, 2, 3\}$. Now let X_1 be the $\{a_3, a_4, b_1, b_2\}$-color switching and update R to be the indices of columns in the red cycle containing the a_1-column. Define z_2 to be the smallest integer in R such that z_2+1 is not in R. Then let X_2 be the $\{a_{z_2}, a_{z_2+1}, b_2, b_3\}$-color switching and update R to include the new column indices of the columns in the red cycle containing the a_1-column. In general, choose z_{2i} or z_{2i+1} to be the smallest integer in the current R so that $z_{2i}+1$ or $z_{2i+1}+1$, respectively, is not in R. Then perform the $\{a_{z_{2i}}, a_{z_{2i}+1}, b_2, b_3\}$-color switching or the $\{a_{z_{2i+1}}, a_{z_{2i+1}+1}, b_1, b_2\}$-color switching to update R. Continue until reaching z_{2n_1} and performing the $\{a_{z_{2n_1}}, a_{z_{2n_1}+1}, b_2, b_3\}$-color switching. It is clear that a red hamiltonian cycle has been created. To see we also result in a hamiltonian cycle of color c_1, we make the color switchings in the order: $X_2, X_1, X_4, X_3, \ldots, X_{2n_1}, X_{2n_1-1}$. For each pair X_{2i} and X_{2i-1}, it follows from Fact 2 that, by making the color switching X_{2i}, the existing hamiltonian cycle of color c_1 is separated into two cycles with one being

$$(a_{z_{2i-2}+1}, b_2)(a_{z_{2i-2}+2}, b_2) \cdots (a_{z_{2i}}, b_2)(a_{z_{2i}}, b_3)(a_{z_{2i}-1}, b_3) \cdots (a_{z_{2i-2}+1}, b_3)(a_{z_{2i-2}+1}, b_2),$$

where $z_0 = 2$. By Fact 1, those two cycles are connected to a hamiltonian cycle of color c_1 again by making the color switching X_{2i-1}. Therefore, we obtain a hamiltonian decomposition of D_k. ∎

In the following discussions, for $m = 2h+1$, we call a vertical edge $(a_i, b_j)(a_i, b_{j+1})$ odd if j is an odd integer between 1 and $2h-1$, and even if j is even.

Lemma 7. *Let $m = 2h+1 \geqslant 5$, n be odd, and $k \geqslant 2$. Assume each H_j in D_k consists of t_j cycles with $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_{k-1}$. If $t_1 = m$ and the sets $K_1 = \{1, 2, 3\}$ and $K_j = \{\pi_j(i) \mid 1 \leqslant i \leqslant 5\}$ for $2 \leqslant j \leqslant k-1$ are mutually disjoint except for K_1 and K_2 having at most one common element $\pi_2(1) = 3$, then D_k has a hamiltonian decomposition.*

Proof. We proceed in a way similar to the proof of Lemma 6. Again, we may assume that each H_j consists of $t_j = gcd(r_j, m) = 2t'_j + 1 \geqslant 3$ cycles for $1 \leqslant j \leqslant k-1$. Since $t_1 = m$, we must have $r_1 = 0$ or $r_1 = m$ which implies that $H_1 \cup F = C_1 \times_{r_1} C_2 = C_1 \times C_2$. We first apply Proposition 2 to $H_1 \cup F = C_1 \times C_2$ with $x = 0$ to obtain a hamiltonian cycle of color c_1 and connect the red cycles in the a_i-columns for $1 \leqslant i \leqslant 3$ to a single red cycle Q_1. In the a_1-column, note that the edges of color c_1 form a matching consisting of all odd vertical edges, i.e., $M = \{(a_1, b_{2i-1})(a_1, b_{2i}) \mid 1 \leqslant i \leqslant h\}$, and all odd edges are red in the a_3-column. For each $2 \leqslant j \leqslant k-1$, we call the a_{l_j}-column *leftmost*, where $l_j = min\, K_j$. Since $K_j \cap (K_1 \cup K_2) = \varnothing$ for $3 \leqslant j \leqslant k-1$ and the sets $K_3, K_4, ..., K_{k-1}$ are mutually disjoint, for each $2 \leqslant j \leqslant k-1$ we apply Lemma 1 to $H_j \cup F = C_1^j \times_{r_j} C_2$, with x being chosen properly, so that for $j = 2$ and $\pi_2(1) = 3$ we use only odd edges in the a_3-column and for $j \geqslant 3$ or $\pi_2(1) > 3$ we can locate two unused adjacent red edges anywhere as we wish in the leftmost a_{l_j}-column, to obtain a hamiltonian cycle of color c_j and connect the involved columns to a single red cycle Q_j. Clearly, for $\pi_2(1) = 3$, Q_1 is connected to Q_2 via the a_3-column. Note that each column is in a single red cycle and each red cycle consists of 3 columns, 5 columns, or 7 columns. We now have $2n_1 + 1$ red cycles for some $n_1 > 0$ as n is odd. Since $m \geqslant 5$, each a_i-column for $3 \leqslant i \leqslant n$ has at least one red odd edge and each a_i-column for $4 \leqslant i \leqslant n$ has property P: for any $1 \leqslant f \leqslant m$ one of the vertical edges $(a_i, b_f)(a_i, b_{f+1})$ and $(a_i, b_{f+2})(a_i, b_{f+3})$ is red (see Fig. 2). Similar to the proof of Lemma 6, we will make some additional edge-disjoint color switchings between red edges and edges of color c_1 so that we obtain a red hamiltonian cycle and a hamiltonian cycle of color c_1 without destroying the existing hamiltonian cycles of color c_j for $2 \leqslant j \leqslant k-1$. For that purpose, we focus on $H_1 \cup F = C_1 \times C_2$. Let $0 < z_1 < z_2 < \cdots < z_{2n_1}$ be the integer sequence defined recursively as follows. Let R be the indices of the columns in the red cycle containing the a_1-column. Having chosen the integers $z_1, z_2, ..., z_i$ and updated R, we define z_{i+1} to be the smallest integer in R such that $z_{i+1} + 1$ is not in R and update R to include all indices of the columns in the red cycles containing the a_1-column and the a_{z_j+1}-columns for $1 \leqslant j \leqslant i+1$. Then $z_1 \geqslant 3$. From the choice of the z_j's it is easy to see that for $1 \leqslant i \leqslant 2n_1$ each a_{z_i+1}-column is either a leftmost column of some Q_j or a free column, namely, all vertical edges in that column are red, so we can locate two adjacent red edges anywhere as we wish in the a_{z_i+1}-column. Next, we define edge-disjoint color switchings $X_1, X_2, ..., X_{2n_1}$ between red edges and edges of color c_1 in $H_1 \cup F = C_1 \times C_2$ so that each X_i is between the a_{z_i}-column and the a_{z_i+1}-column. For $1 \leqslant i \leqslant n_1$, having $X_1, X_2, ..., X_{2i-3}, X_{2i-2}$ defined with specified property, we define the X_{2i-1} and X_{2i} as follows: Let $(a_{z_{2i-1}}, b_{y_i})(a_{z_{2i-1}}, b_{y_i+1})$ be a red odd edge in the $a_{z_{2i-1}}$-column, where y_i is odd. Since the $a_{z_{2i}}$-column has property P, one of the vertical edges $e_1=(a_{z_{2i}}, b_{y_i-1})(a_{z_{2i}}, b_{y_i})$ and $e_2=(a_{z_{2i}}, b_{y_i+1})(a_{z_{2i}}, b_{y_i+2})$ is red, say e_1. Recall that both the $a_{z_{2i-1}+1}$-column (it might happen that $z_{2i-1}+1=z_{2i}$) and the $a_{z_{2i}+1}$-column are either free or leftmost, we define X_{2i-1} to be the $\{a_{z_{2i-1}}, a_{z_{2i-1}+1}, b_{y_i}, b_{y_i+1}\}$-color switching and X_{2i} to be the $\{a_{z_{2i}}, a_{z_{2i}+1}, b_{y_i-1}, b_{y_i}\}$-color switching. Since X_{2i-1} uses only odd vertical edges while X_{2j} does not use odd vertical edges, we conclude that $X_1, X_2, ..., X_{2n_1}$ are edge-disjoint. Now, from the choice of the z_j's it follows that by making the color switchings $X_1, X_2, ..., X_{2n_1}$, we obtain a red hamiltonian cycle. To see that the edges of color c_1 still form a hamiltonian cycle, we make the color switchings in the order $X_1, X_2, ..., X_{2n_1}$ with one pair X_{2i-1} and X_{2i} at each time. It follows from Fact 2 that, by making the color switching X_{2i-1}, the existing hamiltonian cycle of color c_1 is separated into two cycles with one being

$$(a_{z_{2i-1}+1}, b_{y_i})(a_{z_{2i-1}+2}, b_{y_i})\cdots(a_n, b_{y_i})(a_1, b_{y_i})(a_1, b_{y_i+1})$$
$$(a_n, b_{y_i+1})(a_{n-1}, b_{y_i+1})\cdots(a_{z_{2i-1}+1}, b_{y_i+1})(a_{z_{2i-1}+1}, b_{y_i}).$$

Then it follows from Fact 1 that by making the color switching X_{2i} those two cycles are connected to a hamiltonian cycle of color c_1 again. Thus, together with the existing monochromatic hamiltonian cycles of color c_j for $2\leqslant j\leqslant k-1$, we obtain a hamiltonian decomposition of D_k. ∎

We are ready to prove our main result.

Proof of Theorem 1. We proceed by induction on k. For $k=1$, the result is trivial. For $k=2$ or 3, the result follows from Theorems B and C. Assume the result for $k<h$ (with $h\geqslant 4$). Consider $k=h\geqslant 4$. Since $|A|$ is odd, Lagrange's Theorem implies that each $ord(s_i)$ is odd. If $ord(s_i)\leqslant 7$ for all $1\leqslant i\leqslant k$, then each $ord(s_i)$ is a prime. Since S is minimal, $A^{(i)}\cap J_{i+1}=\{0\}$ for $1\leqslant i\leqslant k-1$, where $A^{(i)}$ is the subgroup generated by $S_i=\{s_1, s_2, ..., s_i\}$ and $J_{i+1}=\langle s_{i+1}\rangle$. Then it follows from Proposition 1 and Theorem A that $cay(A, S)$ has a hamiltonian decomposition. Thus, we assume one of the elements in S has odd order at least 9, say s_k. Let $J=\langle s_k\rangle$ and $A_1=A/J$. Then $\bar{S}=\{\bar{s}_1, \bar{s}_2, ..., \bar{s}_{k-1}\}$ is a minimal generating set of A_1 since S is a minimal generating set of A. Also, by Lagrange's Theorem, we know that both $n=|A_1|$ and $m=|J|\geqslant 9$ are odd. Let $A_1=\{\bar{a}_1, \bar{a}_2, ..., \bar{a}_n\}$. By the induction hypothesis, $cay(A_1, \bar{S})$ can be decomposed into $k-1$ hamiltonian cycles $\bar{H}_i=\bar{a}_{\pi_i(1)}\bar{a}_{\pi_i(2)}\cdots\bar{a}_{\pi_i(n)}\bar{a}_{\pi_i(1)}$ for $1\leqslant i\leqslant k-1$, where each π_i is a permutation of $\{1, 2, ..., n\}$. For simplicity, we assume $\pi_1=I$. By Lemma 5, $cay(A, S)\cong D_k$ with each H_j being the 2-factor lifted by $\bar{H}_j$ and F being the 2-factor generated by s_k. Without loss of generality, we assume that each H_j consists of t_j cycles satisfying $t_1\geqslant t_2\geqslant\cdots\geqslant t_{k-1}$. Since $S'=S-\{s_k\}$ can not generate A, it follows from Remark 1 that $cay(A, S')$ is disconnected which implies that each $t_j\geqslant 2$ as

$H_j \subseteq cay(A, S')$. By Lemma 2, $|A_1| \geqslant 3^{k-1}$. Thus, $|A_1| \geqslant 27$ for $k=4$, and $|A_1| > 5[5(k-1)-7]$ for $k \geqslant 5$. By applying Lemmas 3 and 4 to $cay(A_1, \bar{S})$, we may assume that all π_i, for $1 \leqslant i \leqslant k-1$, satisfy the conditions in Lemmas 6 and 7. Now, it follows from Lemmas 6 and 7 that $cay(A, S) \cong D_k$ has a hamiltonian decomposition. ∎

I believe that the techniques used here for abelian groups of odd order are useful in obtaining a similar result for abelian groups of even order.

Acknowledgments

The author is grateful to the referees whose many comments have significantly helped to improve the presentation of this article. In particular, the author thanks the referees for providing a proof for Lemma 4.

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